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On a Problem of G. G. Lorentz

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DEDICATED TO GEORGE LORENTZ ON HIS 90TH BIRTHDAY

Let $C(B)$ denote the space of real-valued continuous functions on B. At the conference "Harmonic Analysis and Approximations" held at Nor Amberd in Armenia in September, 1998, the following general problem was posed by Professor George G. Lorentz: Find conditions on finite-dimensional subspaces U and V so that U/V is a uniqueness set in the problem of best uniform approximation to elements of $C(B)$. In this paper we consider this problem.

1. INTRODUCTION

We first set some notation. For ease of exposition, assume B is a finite union of connected compact components, none of which is a singleton, in $\mathbb{R}^d$ for some d. Let $C(B)$ denote the space of real-valued continuous functions on B. Let U_n and V_m be n and m-dimensional linear subspaces of $C(B)$, respectively. We further assume that V_m contains a function which is strictly positive on B. Set

$$\frac{U_n}{V_m} = \left\{ \frac{u}{v} : u \in U_n,\, v \in V_m,\, v > 0 \right\}.$$

We are interested in the problem of when, to each f in $C(B)$, we have at most one best approximant from the set U_n/V_m in the uniform norm on B. This problem was posed by G. G. Lorentz in his talk at the conference "Harmonic Analysis and Approximations" at Nor Amberd in Armenia in September, 1998.

Before stating our main results let us recall some facts concerning "generalized rational approximation." In this setting it is not necessary that a best approximant from U_n/V_m to each f in $C(B)$ exist. However, it is always possible to characterize a best approximant if it does in fact exist.

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THEOREM 1.1 (Cheney and Loeb [5], Cheney [4]). *Let $f \in C(B)$. A necessary and sufficient condition for an element r^* to be a best approximant to f from U_n/V_m is that the zero function be a best approximant to $f - r^*$ from the linear space $U_n + r^* V_m$ given by*

$$U_n + r^* V_m = \{u + r^* v : u \in U_n, v \in V_m\}.$$

We are interested in the question of uniqueness of the best approximant, if it exists. On approximating from linear spaces (of finite dimension) the question of uniqueness, in the uniform norm, was considered by Haar [7]. He proved that for a k-dimensional approximating subspace W_k of $C(B)$, the best approximant (which always exists) to each element of $C(B)$ is unique iff there does not exist a nontrivial $w \in W_k$ which vanishes at k or more distinct points in B. As is more or less standard, we call linear spaces which have this property *Haar spaces*. (When $B = [a, b]$ the term *Chebyshev space* is more commonly used.)

Theorem 1.1 has the following consequence.

PROPOSITION 1.2 (Cheney [3], Cheney [4]). *If r^* is a best approximant to $f \in C(B)$ from U_n/V_m and if $U_n + r^* V_m$ is a Haar space, then r^* is the unique best approximant to f from U_n/V_m.*

Thus a necessary condition that U_n/V_m be a *uniqueness set*, i.e., each $f \in C(B)$ have at most one best approximant from U_n/V_m, is that $U_n + r^* V_m$ be a Haar space for every $r^* \in U_n/V_m$. The converse result need not quite hold in this generality only because our approximating set has the restriction that we only consider u/v where v is strictly of one sign on B.

Uniqueness of rational approximants is known in two cases. If U_n and V_m are algebraic polynomials of degree $n-1$ and $m-1$, respectively, and $B = [a, b]$, then uniqueness is due to Achieser [1] (see the more accessible Achieser [2]). If U_n and V_m are the analogous trigonometric polynomials then uniqueness, within the class of 2π-periodic continuous functions, was recently proved in Lorentz *et al.* [8, p. 217].

Before considering conditions under which $U_n + r^* V_m$ is a Haar space for every r^* in U_n/V_m, let us first note some facts concerning the subspaces $U_n + r^* V_m$ and some necessary properties which are implied by the assumption that $U_n + r^* V_m$ is a Haar space for every r^* in U_n/V_m.

LEMMA 1.3. *For each $r^* \in U_n/V_m$,*

$$n \leqslant \dim(U_n + r^* V_m) \leqslant n + m - 1, \tag{1.1}$$

and if U_n and V_m are Haar spaces, and $r^* \neq 0$, then we also have

$$m \leqslant \dim(U_n + r^* V_m).$$

Proof. The lower bound in (1.1) is a consequence of the fact that $U_n \subseteq U_n + r^* V_m$ for any $r^* \in U_n/V_m$. The perhaps somewhat surprising upper bound may be proven as follows. Let

$$U_n = \operatorname{span}\{u_1, ..., u_n\}, \qquad V_m = \operatorname{span}\{v_1, ..., v_m\},$$

and $r^* = u^*/v^*$, where $u^* = \sum_{i=1}^{n} a_i^* u_i$ and $v^* = \sum_{i=1}^{m} b_i^* v_i$, $v^* > 0$ on B. Then

$$U_n + r^* V_m = \operatorname{span}\{u_1, ..., u_n, r^* v_1, ..., r^* v_m\}.$$

The above $n+m$ functions which span $U_n + r^* V_m$ are linearly dependent since

$$\sum_{i=1}^{n} a_i^* u_i - r^* \left(\sum_{i=1}^{m} b_i^* v_i \right) = u^* - \left(\frac{u^*}{v^*} \right) v^* = 0.$$

Thus

$$\dim(U_n + r^* V_m) \leqslant n + m - 1.$$

Assume U_n and V_m are Haar spaces. This implies (since B contains a continuum of points) that if $u \in U_n$, $v \in V_m$, and $uv = 0$, then $u = 0$ or $v = 0$. From this property it follows that for $r^* \in U_n/V_m$, $r^* \neq 0$,

$$m = \dim(r^* v_1, ..., r^* v_m) = \dim(r^* V_m) \leqslant \dim(U_n + r^* V_m). \quad \blacksquare$$

PROPOSITION 1.4. *If $U_n + r^* V_m$ is a Haar space for every $r^* \in U_n/V_m$, then U_n and V_m are themselves Haar spaces. The converse result holds if $n = 1$ or $m = 1$.*

Proof. We set $r^* = 0$ in $U_n + r^* V_m$ and deduce that U_n is a Haar space.

Assume V_m is not a Haar space. There then exists a $\tilde{v} \in V_m$, $\tilde{v} \neq 0$, that vanishes at at least m distinct points of B. Given any $n-1$ points distinct from the above m points, there exists a $u^* \in U_n$, $u^* \neq 0$, which vanishes thereon. Let $v^* \in V_m$ be strictly positive on B, and set $r^* = u^*/v^*$. Then

$$r^* \tilde{v} \in r^* V_m \subseteq U_n + r^* V_m,$$

and $r^* \tilde{v} \neq 0$ has at least $n+m-1$ distinct zeros. Since $\dim(U_n + r^* V_m) \leqslant n + m - 1$, this contradicts the assumption that $U_n + r^* V_m$ is a Haar space.

Assume U_n and V_m are Haar spaces and $n = 1$ or $m = 1$. If $r^* = 0$, then $U_n + r^* V_m = U_n$ is a Haar space. For $r^* \neq 0$ it follows from Lemma 1.3 that

$$\max\{n, m\} = \dim\{U_n + r^* V_m\}.$$

If $m=1$, then $U_n+r^*V_1=U_n$ is a Haar space. If $n=1$ (and $r^*\neq 0$) then r^* is strictly of one sign on B (U_1 is a Haar space) and $U_1+r^*V_m=r^*V_m$ is also a Haar space. ∎

The converse result need not hold in general. That is, it may be that U_n and V_m are Haar spaces, while $U_n+r^*V_m$ is not a Haar space for some r^* in U_n/V_m, n, $m\geqslant 2$. An example thereof may be found in Cheney [4, p. 169].

Set

$$U_nV_m=\{uv : u\in U_n, v\in V_m\}.$$

(By uv we mean simple multiplication, i.e., $(uv)(x)=u(x)\,v(x)$.) Under relatively mild assumptions on U_n and V_m it may easily be shown that

$$\dim(U_nV_m)\geqslant n+m-1.$$

We will prove that if $\dim(U_nV_m)=n+m-1$ and U_n, V_m are Haar spaces, then

$$U_n+r^*V_m$$

is a Haar space for every r^* in U_n/V_m. This is one of the main result of this paper.

THEOREM 1.5. *If U_n and V_m are Haar spaces in $C(B)$ and*

$$\dim(U_nV_m)=n+m-1$$

then U_n/V_m is a uniqueness set (and U_nV_m is a Haar space).

This condition implying uniqueness (and also existence) is neither fortuitous nor unexpected. Consider $r^*=u^*/v^*\in U_n/V_m$. Then

$$u+r^*v=\frac{uv^*+u^*v}{v^*}.$$

By assumption v^* does not vanish on B. Thus the zero sets of $u+r^*v$ and uv^*+u^*v are identical and the latter function is contained in U_nV_m. Since we are interested in conditions implying that $U_n+r^*V_m$ is a Haar space, and since $\dim(U_n+r^*V_m)\leqslant n+m-1$, it is thus natural to consider when U_nV_m is a Haar space of dimension $n+m-1$.

Note that sums of the form uv^*+u^*v are a manifold within U_nV_m. The two restrictions are that v^* be strictly of one sign on B, and that we only permit the sum of two products (rather than $\min\{n, m\}$ products which are in general necessary to span U_nV_m).

The demand that $\dim(U_n V_m) = n + m - 1$ is very stringent. Theorem 1.5 is a consequence of the following. Assume that U_n and V_m have the property that

$$B = \overline{\text{supp}\{g\}} \tag{1.2}$$

for every nonzero g in U_n or V_m ($\text{supp}\{g\} = \{x : g(x) \neq 0\}$). This property holds in particular if U_n and V_m are Haar spaces.

THEOREM 1.6. *Let U_n and V_m be n- and m-dimensional subspaces of $C(B)$, n, $m \geqslant 2$. Assume that* (1.2) *holds. Then* $\dim(U_n V_m) \geqslant n + m - 1$. *Furthermore* $\dim(U_n V_m) = n + m - 1$ *if and only if there exist w_1, $w_2 \in C(B)$ and a function h defined on B such that*

$$U_n = \text{span}\{w_1 h^{i-1} : i = 1, ..., n\} \tag{1.3}$$

$$V_m = \text{span}\{w_2 h^{i-1} : i = 1, ..., m\} \tag{1.4}$$

Remark. In the statement of Theorem 1.6 we do not demand that h be continuous. We will show, by example, that h need not be continuous and sometimes cannot possibly be continuous. However as $w_1 h^{i-1} \in U_n \subset C(B)$, for $i = 1, ..., n$, and $w_2 h^{i-1} \in V_m \subset C(B)$, for $i = 1, ..., m$, the function h must be continuous at those points where either w_1 or w_2 is nonzero.

Theorem 1.6, where $U_n = V_m$, was proven by Granovsky [6]. In fact he had fewer restrictions on both B and U_n. His motivation for considering this problem came from questions in mathematical statistics connected with regression functions and the theory of experimental design.

2. PROOF OF THEOREM 1.6

If (1.3) and (1.4) hold, then $\dim(U_n V_m) = n + m - 1$. It is the converse direction which we must labour to prove. Since that proof is somewhat lengthy, we will divide it into a series of steps.

Before embarking on these steps, let us note that Theorem 1.6 is not valid without some conditions on U_n and V_m. For example, assume $U_3 = V_3 = \text{span}\{1, x, |x|\}$ on $[-1, 1]$. Then $\dim(U_3 U_3) = 5$ and yet U_3 is not of the desired form. Similarly if U_n contains n functions with disjoint support, then $\dim(U_n U_n) = n$. (Note that in both examples there exist nonzero u_1, $u_2 \in U_n$ for which $u_1 u_2 = 0$, see Granovsky [6].)

In the proof of Theorem 1.6 we follow, with some modifications, the basic form of the proof as given in Granovsky [6].

Let us assume that $n \geqslant m \geqslant 2$. We start by choosing distinct points $x_1, ..., x_n$ in B for which

$$\dim U_n|_{\{x_1, ..., x_n\}} = n \tag{2.1}$$

and such that

$$\dim V_m|_{\{x_{i_1}, ..., x_{i_m}\}} = m \tag{2.2}$$

for every choice of distinct $\{i_1, ..., i_m\} \subseteq \{1, ..., n\}$. That such a choice of $\{x_1, ..., x_n\}$ exists follows from our assumption concerning the form of B, U_n and V_m.

Let $p_i \in U_n$, $i = 1, ..., n$, satisfy

$$p_i(x_k) = \delta_{ik}, \qquad i, k = 1, ..., n. \tag{2.3}$$

Such p_i exist and form a basis for U_n as a consequence of (2.1). Let $q_j \in V_m$, $j = 1, ..., m$, satisfy

$$q_j(x_k) = \delta_{jk}, \qquad j, k = 1, ..., m. \tag{2.4}$$

From (2.2) such q_j exist and form a basis for V_m. Furthermore from (2.2),

$$q_j(x_k) \neq 0, \qquad j = 1, ..., m, \quad k = m+1, ..., n. \tag{2.5}$$

LEMMA 2.1. *The* $n+m-1$ *functions* $p_1 q_1, p_2 q_2, ..., p_m q_m, p_2 q_1, ..., p_n q_1$ *are linearly independent.*

Proof. Assume

$$\sum_{j=1}^{m} \beta_j p_j q_j + \left(\sum_{i=2}^{n} \alpha_i p_i \right) q_1 = 0. \tag{2.6}$$

Evaluate the left-hand side of (2.6) at the points x_k, $k = 1, ..., m$, to obtain (using (2.3) and (2.4))

$$\beta_k = 0, \qquad k = 1, ..., m.$$

Thus (2.6) reduces to

$$\left(\sum_{i=2}^{n} \alpha_i p_i \right) q_1 = 0.$$

From our assumption (1.2) on B, U_n, and V_m, if $uv = 0$, $u \in U_n$, $v \in V_m$, then $u = 0$ or $v = 0$. Thus either $q_1 = 0$ (which is simply not true), or

$$\sum_{i=2}^{n} \alpha_i p_i = 0.$$

As the $p_2, ..., p_n$ are linearly independent, this implies that $\alpha_2 = \cdots = \alpha_n = 0$. ∎

From Lemma 2.1 it follows that $\dim(U_n V_m) \geqslant n+m-1$. Since

$$U_n V_m = \operatorname{span}\{p_k q_\ell : k=1, ..., n, \ell=1, ..., m\}$$

Lemma 2.1 further implies that $\dim(U_n V_m) = n+m-1$ if and only if each $p_k q_\ell$ is a linear combination of the $n+m-1$ functions in the statement of Lemma 2.1, i.e., $p_i q_j$ satisfying $i=j$ or $j=1$. We assume in what follows in this section that $\dim(U_n V_m) = n+m-1$.

LEMMA 2.2. *For each k, ℓ, s, satisfying $k \neq \ell$, $k=1, ..., n$, ℓ, $s=1, ..., m$, there exist constants $\alpha^s_{k,\ell}$, $\beta^s_{k,\ell}$ for which*

$$p_k q_\ell = (\alpha^s_{k,\ell} p_k + \beta^s_{k,\ell} p_\ell)\, q_s. \tag{2.7}$$

Proof. It suffices to prove (2.7) for $s=1$. (The choice of q_1 here and in Lemma 2.1 is arbitrary and it may be replaced by q_s for any $s=1, ..., m$.)

From Lemma 2.1 we have (since $\dim(U_n V_m) = n+m-1$),

$$p_k q_\ell = \left(\sum_{i=2}^{n} \gamma^i_{k,\ell} p_i \right) q_1 + \sum_{j=1}^{m} \sigma^j_{k,\ell} p_j q_j, \tag{2.8}$$

for some constants $\gamma^i_{k,\ell}$ and $\sigma^j_{k,\ell}$.

Assume $k \neq \ell$. We will first evaluate (2.8) at x_r, $r=1, ..., m$. Since $k \neq \ell$,

$$p_k(x_r)\, q_\ell(x_r) = 0, \qquad r=1, ..., m$$

and

$$p_i(x_r)\, q_1(x_r) = 0, \qquad r=1, ..., m,$$

for each $i=2, ..., n$. Furthermore

$$p_j(x_r)\, q_j(x_r) = \delta_{rj}, \qquad r, j=1, ..., m.$$

Thus from (2.8) it follows that

$$\sigma^r_{k,\ell} = 0, \qquad r=1, ..., m,$$

and (2.8) reduces to

$$p_k q_\ell = \left(\sum_{i=2}^{n} \gamma^i_{k,\ell} p_i \right) q_1. \tag{2.9}$$

We now evaluate (2.9) at x_r, $r=m+1, ..., n$. By construction

$$p_k(x_r)\, q_\ell(x_r) = \delta_{kr} q_\ell(x_r),$$

and from (2.5), $q_\ell(x_r) \neq 0$ for $r = m+1, ..., n$. Similarly

$$\left(\sum_{i=2}^{n} \gamma_{k,\ell}^{i} p_i \right)(x_r) = \gamma_{k,\ell}^{r},$$

(while $q_1(x_r) \neq 0$). Thus for $k \neq r$, $r = m+1, ..., n$, we have

$$\gamma_{k,\ell}^{r} = 0.$$

We may therefore rewrite (2.9) as

$$p_k q_\ell = \left(\sum_{i=2}^{m} \gamma_{k,\ell}^{i} p_i \right) q_1, \tag{2.10}$$

for $k \neq \ell$, and $k, \ell = 1, ..., m$, and

$$p_k q_\ell = \left(\sum_{i=2}^{m} \gamma_{k,\ell}^{i} p_i + \gamma_{k,\ell}^{k} p_k \right) q_1, \tag{2.11}$$

if $k = m+1, ..., n$. (Note that if $m = 2$, then (2.10) and (2.11) are of the desired form.)

Choose $s \in \{1, ..., m\}$, $s \neq \ell$. We can replace q_ℓ by q_s in (2.10) and (2.11) to obtain

$$p_k q_s = \left(\sum_{i=2}^{m} \gamma_{k,s}^{i} p_i \right) q_1, \tag{2.12}$$

for $k \neq s$, and $k = 1, ..., m$, and

$$p_k q_s = \left(\sum_{i=2}^{m} \gamma_{k,s}^{i} p_i + \gamma_{k,s}^{k} p_k \right) q_1, \tag{2.13}$$

if $k = m+1, ..., n$.

Multiplying (2.10) by q_s and (2.12) by q_ℓ, it follows that for k, ℓ, s distinct in $\{1, ..., m\}$

$$\left(\sum_{i=2}^{m} \gamma_{k,\ell}^{i} p_i \right) q_1 q_s = \left(\sum_{i=2}^{m} \gamma_{k,s}^{i} p_i \right) q_1 q_\ell.$$

From our assumption (1.2) this implies

$$\left(\sum_{i=2}^{m} \gamma_{k,\ell}^{i} p_i \right) q_s = \left(\sum_{i=2}^{m} \gamma_{k,s}^{i} p_i \right) q_\ell. \tag{2.14}$$

Evaluate (2.14) at x_s. As $q_s(x_s)=1$ and $q_\ell(x_s)=0$, we obtain

$$\sum_{i=2}^{m} \gamma_{k,\ell}^{i} p_i(x_s)=0.$$

Since $p_i(x_s)=\delta_{is}$, this implies that $\gamma_{k,\ell}^{s}=0$. This is valid for every s distinct from k and ℓ in $\{1, ..., m\}$. This proves (2.7).

We now assume that $k\in\{m+1, ..., n\}$. We multiply (2.11) by q_s and (2.13) by q_ℓ, $s\neq\ell$, and parallel the above analysis to again obtain (2.7). ∎

LEMMA 2.3. *Fix k, ℓ in (2.7). The coefficients $(\alpha_{k,\ell}^{s}, \beta_{k,\ell}^{s})$, $s=1, ..., m$, are pairwise linearly independent, i.e., $(\alpha_{k,\ell}^{s}, \beta_{k,\ell}^{s})\neq\gamma(\alpha_{k,\ell}^{r}, \beta_{k,\ell}^{r})$ for any constant γ and distinct s, r in $\{1, ..., m\}$.*

Proof. Assume to the contrary that

$$(\alpha_{k,\ell}^{s}, \beta_{k,\ell}^{s})=\gamma(\alpha_{k,\ell}^{r}, \beta_{k,\ell}^{r})$$

for some γ and $s\neq r$. Then from (2.7)

$$p_k q_\ell=\gamma(\alpha_{k,\ell}^{r} p_k+\beta_{k,\ell}^{r} p_\ell)\, q_s=(\alpha_{k,\ell}^{r} p_k+\beta_{k,\ell}^{r} p_\ell)\, q_r.$$

This implies that either

$$\gamma q_s-q_r=0$$

or

$$\alpha_{k,\ell}^{r} p_k+\beta_{k,\ell}^{r} p_\ell=0.$$

The first option is invalid since the q_k are linearly independent. The second equation together with (2.7) implies that $p_k q_\ell=0$, which again is impossible. ∎

We will fix the k, ℓ in (2.7). For convenience in what follows assume k, $\ell\in\{1, ..., m\}$, $k\neq\ell$, and set

$$Z_{k\ell}=\{x : p_k(x)\, q_\ell(x)=0\}.$$

Note that by our assumption (1.2) $\overline{B\backslash Z_{k\ell}}=B$. From (2.7) it follows that if $x\notin Z_{k\ell}$, then in addition to $p_k(x)$, $q_\ell(x)$ not vanishing we also have $q_s(x)\neq 0$ and $(\alpha_{k,\ell}^{s} p_k+\beta_{k,\ell}^{s} p_\ell)(x)\neq 0$ for each $s=1, ..., m$. On $B\backslash Z_{k\ell}$, set

$$h(x)=\frac{p_\ell(x)}{p_k(x)}, \tag{2.15}$$

$$\Delta_s(x)=\alpha_{k,\ell}^{s}+\beta_{k,\ell}^{s} h(x), \quad s=1, ..., m, \tag{2.16}$$

and

$$w_2(x)=\frac{q_\ell(x)}{\prod_{s=1}^m \Delta_s(x)}. \tag{2.17}$$

From (2.7) we have

$$q_s(x)=\frac{q_\ell(x)}{\Delta_s(x)}. \tag{2.18}$$

Each of these quantities is well defined and continuous on $B\backslash Z_{k\ell}$. In fact neither $\Delta_s(x)$ nor $w_2(x)$ vanish on $B\backslash Z_{k\ell}$.

LEMMA 2.4. *We have*

$$V_m=\operatorname{span}\{w_2 h^{i-1} : i=1, ..., m\}.$$

Proof. We first restrict ourselves to $B\backslash Z_{k\ell}$. If $v\in V_m$, then since the $\{q_s\}_{s=1}^m$ span V_m and from (2.17) and (2.18)

$$\begin{aligned} v(x) &= \sum_{s=1}^m a_s q_s(x) = \sum_{s=1}^m a_s \frac{q_\ell(x)}{\Delta_s(x)} \\ &= \frac{q_\ell(x)}{\prod_{s=1}^m \Delta_s(x)} \sum_{s=1}^m \alpha_s \left(\prod_{\substack{r=1\\ r\neq s}}^m \Delta_r(x)\right) \\ &= w_2(x) \sum_{s=1}^m a_s \left(\prod_{\substack{r=1\\ r\neq s}}^s \Delta_r(x)\right), \end{aligned}$$

for $x\in B\backslash Z_{k\ell}$. The expression

$$\prod_{\substack{r=1\\ r\neq s}}^m \Delta_r(x) = \prod_{\substack{r=1\\ r\neq s}}^m (\alpha_{k,\ell}^r + \beta_{k,\ell}^r h(x))$$

is a polynomial of degree at most $m-1$ in h. Thus on $B\backslash Z_{k\ell}$

$$v = w_2 \sum_{i=1}^m \gamma_i h^{i-1}.$$

It is easily shown, using Lemma 2.3, that

$$\operatorname{span}\left\{ \prod_{\substack{r=1\\ r\neq s}}^m \Delta_r(x) : s=1, ..., m\right\} = \operatorname{span}\{h^{i-1} : i=1, ..., m\}.$$

Thus the $w_2 h^{i-1}$, $i=1, ..., m$, span V_m on $B \backslash Z_{k\ell}$. These functions are linearly independent on $B \backslash Z_{k\ell}$ since otherwise there exists a nontrivial element of V_m which identically vanishes on $B \backslash Z_{k\ell}$. This contradicts (1.2).

Since each $w_2 h^{i-1} \in V_m$ and $V_m \subset C(B)$, it follows from our assumption (1.2) that each of these functions can be uniquely extended from $B \backslash Z_{k\ell}$ to B as elements in $C(B)$. This proves the lemma. ▮

The function w_2 is well defined on all of B. It is, after all, a function in $C(B)$ and V_m. This is not true of h, which is a ratio of two functions in $C(B)$ (and V_m). The function h is continuous at every point where w_2 does not vanish, which includes $B \backslash Z_{k\ell}$, but it need not be continuous on all of B. Nonetheless, since $w_2 h^{i-1}$ is continuous on all of B, this restricts the permissible types of discontinuities of h.

What we have done for V_m we can also do for U_n.

LEMMA 2.5. *For each k, ℓ, s, satisfying $k \neq \ell$, k, $\ell = 1, ..., m$, $s = 1, ..., n$, there exist constants $\gamma^s_{k,\ell}$, $\sigma^s_{k,\ell}$ for which*

$$p_k q_\ell = p_s(\gamma^s_{k,\ell} q_k + \sigma^s_{k,\ell} q_\ell). \tag{2.19}$$

Furthermore the coefficients $(\gamma^s_{k,\ell}, \sigma^s_{k,\ell})$, $s = 1, ..., n$, are pairwise linearly independent.

Proof. Rather than parallel our previous analysis we will show how (2.19) follows from (2.7).

We recall that (2.7) has the form

$$p_k q_\ell = (\alpha^s_{k,\ell} p_k + \beta^s_{k,\ell} p_\ell)\, q_s,$$

for $k \neq \ell$, $k = 1, ..., n$, ℓ, $s = 1, ..., m$. We can rewrite this as

$$\beta^s_{k,\ell} p_\ell q_s = p_k (q_\ell - \alpha^s_{k,\ell} q_s).$$

Note that $\beta^\ell_{k,\ell} = 0$ and thus $\beta^s_{k,\ell} \neq 0$ for $s \neq \ell$. As such we have

$$p_\ell q_s = p_k (\gamma^k_{\ell,s} q_\ell + \sigma^k_{\ell,s} q_s)$$

for all k, ℓ, s satisfying $k = 1, ..., n$, $k \neq \ell$, and $s \neq \ell$. For $k = \ell$, set $\gamma^k_{\ell,s} = 0$ and $\sigma^k_{\ell,s} = 1$. We now simply rename ℓ, s, and k as k, ℓ, and s, respectively, to obtain (2.19).

Paralleling the proof of Lemma 2.3, it follows that the $(\gamma^s_{k,\ell}, \sigma^s_{k,\ell})$ are pairwise linearly independent, $s = 1, ..., n$. ▮

As previously, on $B \backslash Z_{k\ell}$ (recall that we chose $k, \ell \in \{1, ..., m\}$, $k \neq \ell$) set

$$H(x) = \frac{q_k(x)}{q_\ell(x)},$$

$$\Sigma_s(x) = \gamma^s_{k,\ell} H(x) + \sigma^s_{k,\ell}, \qquad s = 1, ..., n,$$

and

$$W_1(x) = \frac{p_k(x)}{\prod_{s=1}^n \Sigma_s(x)}.$$

From (2.19) we have

$$p_s(x) = \frac{p_k(x)}{\Sigma_s(x)}.$$

Note that from (2.19) and the definition of $Z_{k\ell}$, each of these quantities is well defined and continuous on $B \backslash Z_{k\ell}$. Analogously to Lemma 2.4, we obtain

LEMMA 2.6. *We have*

$$U_n = \operatorname{span}\{W_1 H^{i-1} : i = 1, ..., n\}.$$

Both U_n and V_m, individually, have the desired form. But in one case we have multiplier H and in the other case multiplier h. Our claim in Theorem 1.6 is that they are equal, or to be more precise, that they can be chosen to be equal. This we now prove.

We recall that $h = p_\ell / p_k$ and $H = q_k / q_\ell$. Take (2.7) with $s = k$, i.e.,

$$p_k q_\ell = (\alpha^k_{k,\ell} p_k + \beta^k_{k,\ell} p_\ell)\, q_k.$$

Divide by $p_k q_\ell$ (we restrict ourselves to $B \backslash Z_{k,\ell}$) to obtain

$$1 = (\alpha^k_{k,\ell} + \beta^k_{k,\ell} h)\, H,$$

which implies

$$H = \frac{1}{\alpha^k_{k,\ell} + \beta^k_{k,\ell} h} \left(= \frac{1}{\Delta_k} \right). \tag{2.20}$$

Note that $\beta^k_{k,\ell} \neq 0$. Our desired result will follow from Lemma 2.7, which we state in a rather general form, as we will use it again.

LEMMA 2.7. *Let*

$$W_k = \operatorname{span}\{mg^{i-1} : i = 1, ..., k\} \tag{2.21}$$

be a k-dimensional subspace of $C(B)$, $k \geqslant 2$. *Assume* $B = \overline{\operatorname{span}\{w\}}$ *for every* $w \in W_k$, $w \neq 0$. *Let* $a, b, c, d \in \mathbb{R}$, $ad - bc \neq 0$. *Define the functions*

$$G = \frac{ag+b}{cg+d}, \qquad M = \frac{m(cg+d)^{k-1}}{(ad-bc)^{k-1}}. \tag{2.22}$$

Then

$$W_k = \operatorname{span}\{MG^{i-1} : i = 1, ..., k\}. \tag{2.23}$$

Furthermore, if W_k *can be written in the forms* (2.21) *and* (2.23) *for some choices of* m, M, g, *and* G, *then* (2.22) *holds for some constants* a, b, c, d *satisfying* $ad - bc \neq 0$. *That is, the* m *and* g *of* W_k *are unique up to the above linear fractional transformation.*

Proof. From $G = (ag+b)/(cg+d)$ it follows that $g = (dG-b)/(-cG+a)$ and

$$\begin{aligned} mg^{i-1} &= m\left(\frac{dG-b}{-cG+a}\right)^{i-1} = \frac{m}{(-cG+a)^{k-1}}(dG-b)^{i-1}(-cG+a)^{k-1} \\ &= \frac{m(cg+d)^{k-1}}{(ad-bc)^{k-1}}(dG-b)^{i-1}(-cG+a)^{k-i} \\ &= M(dG-b)^{i-1}(-cG+a)^{k-i}. \end{aligned}$$

Each $(dG-b)^{i-1}(-cG+a)^{k-i}$, $i = 1, ..., k$, is a polynomial in G of degree at most $k-1$. Thus

$$W_k = \operatorname{span}\{mg^{i-1} : i = 1, ..., k\} \subseteq \operatorname{span}\{MG^{i-1} : i = 1, ..., k\}.$$

Since W_k is of dimension k this implies equality, i.e., (2.23) holds.

Assume W_k can be written in the form (2.21) and (2.23) for some choices of m, M, g, and G. Since M, $MG \in C(B)$ we mus have (from (2.21))

$$G = \frac{MG}{M} \in \frac{W_k}{W_k} = \frac{\sum_{i=1}^{k} \alpha_i g^{i-1}}{\sum_{i=1}^{k} \beta_i g^{i-1}} \tag{2.24}$$

on $B \backslash Z$, where Z is the union of the zero sets of M and m. On this set the continuous functions g and G take on a continuum of values (since, for example, the zero set of $m(c-g) \in W_k$ is, by assumption, small for every constant c). As such we may regard the rightmost expression in (2.24) as a rational function in g. We can factor out its common divisors and write it in the form

$$G = \frac{\alpha \prod_{i=1}^{s} (g-\gamma_i)}{\prod_{j=1}^{t} (g-\delta_j)}$$

for $s, t \leqslant k-1$, where $\gamma_i \neq \delta_j$, $i=1, ..., s$, $j=1, ..., t$, and $\alpha \in \mathbb{R}$. We wish to prove that $s, t \leqslant 1$. If $k=2$ we are finished. As such, assume $k>2$. Then

$$G^{k-1} = \frac{MG^{k-1}}{M} \in \frac{W_k}{W_k}$$

and

$$G^{k-1} = \frac{\alpha^{k-1} \prod_{i=1}^{s} (g-\gamma_i)^{k-1}}{\prod_{j=1}^{t} (g-\delta_j)^{k-1}}. \tag{2.25}$$

Since g takes on a continuum of values and the rational function (2.25) is in irreducible form, it follows that this ratio is an element of W_k/W_k if and only if $s, t \leqslant 1$.

Thus

$$G = \frac{ag+b}{cg+d}$$

for some constants a, b, c, d. If $ad-bc=0$, then G is a constant function and (2.23) cannot hold. As such we must have $ad-bc \neq 0$.

We now consider M. Since

$$M \in W_k = \operatorname{span}\{mg^{i-1} : i=1, ..., k\},$$

we have

$$M = m\left(\sum_{i=1}^{k} \alpha_i g^{i-1}\right).$$

Similarly

$$MG^{k-1} = m\left(\sum_{i=1}^{k} \alpha_i g^{i-1}\right)\left(\frac{ag+b}{cg+d}\right)^{k-1} \in W_k$$

and so is also of the form

$$m\left(\sum_{i=1}^{k} \beta_i g^{i-1}\right).$$

Thus

$$\left(\frac{ag+b}{cg+d}\right)^{k-1}=\frac{\sum_{i=1}^{k}\beta_i g^{i-1}}{\sum_{i=1}^{k}\alpha_i g^{i-1}}.$$

Since the left-hand side is irreducible in g, this implies that

$$\sum_{i=1}^{k}\alpha_i g^{i-1}=\alpha(cg+d)^{k-1}$$

for some constant α, $\alpha\neq 0$. Thus

$$M=m\alpha(cg+d)^{k-1}. \quad \blacksquare$$

To conclude the proof of Theorem 1.6 we now simply apply Lemma 2.7 to (2.20).

Remark. Assume

$$W_k=\operatorname{span}\{mg^{i-1}: i=1, ..., k\}$$

as in the statement of Lemma 2.7. We know that g is continuous where m does not vanish. What happens at the zeros of m? If $m(x^*)=0$ then necessarily $(mg^{i-1})(x^*)=0$, $i=1, ..., k-1$. It is, however, possible that $(mg^{k-1})(x^*)\neq 0$, in which case $\lim_{x\to x^*}|g(x^*)|=\infty$.

Before returning to the question of uniqueness in rational approximation, let us consider the following question. Given a k-dimensional subspace W_k of $C(B)$, how can we decide if W_k is of the form

$$W_k=\operatorname{span}\{mg^{i-1}: i=1, ..., k\}$$

for some m and g. (We assume $B=\overline{\operatorname{supp}\{w\}}$ for every $w\in W_k$, $w\neq 0$.) It follows from Theorem 1.6 that W_k has this form if and only if $\dim(W_k W_k)=2k-1$. In the proof of Theorem 1.6 we made use of the functions p_i and q_j. Here they are the same. It then follows from the proof of Theorem 1.6 (see (2.15)–(2.18)) that we can take $g=q_1/q_2$ and $m=q_1/(\prod_{s=1}^{k}\Delta_s)=q_2\cdots q_k/q_1^{k-2}$. As such we have:

Proposition 2.8. *Let W_k be a k-dimensional subspace of $C(B)$, and let $B=\overline{\operatorname{supp}\{w\}}$ for every $w\in W_k$, $w\neq 0$. Let $x_1, ..., x_k$ be distinct points in B for which*

$$\dim W_k|_{\{x_1, ..., x_k\}}=k$$

and let $q_i \in W_k$ *satisfy* $q_i(x_j) = \delta_{ij}$, $i, j = 1, ..., k$. *Then*

$$W_k = \operatorname{span}\{mg^{i-1} : i = 1, ..., k\}$$

for some m *and* g *if and only if*

$$\frac{q_3 \cdots q_k}{q_1^{k-i-1} q_2^{i-2}} \in W_k, \qquad i = 1, ..., k.$$

Finally we note that Theorem 1.6 can be generalized to a product of any finite number of finite-dimensional subspaces.

COROLLARY 2.9. *Let* U^j *be an* n_j*-dimensional subspace of* $C(B)$, $j = 1, ..., r$. *Assume that the properties of* B *and the* U^j *hold as in* Theorem 1.6. *Then* $\dim(U^1 \cdots U^r) \geqslant n_1 + \cdots + n_r - (r-1)$. *Furthermore* $\dim(U^1 \cdots U^r) = n_1 + \cdots + n_r - (r-1)$ *if and only if there exist* $w_j \in C(B)$ *and a function* h *defined on* B *such that*

$$U^j = \operatorname{span}\{w_j h^{i-1} : i = 1, ..., n_j\}, \qquad j = 1, ..., r.$$

3. PROOF OF THEOREM 1.5

In this section we return to a consideration of the problem of uniqueness in approximation from U_n/V_m. We prove Theorem 1.5. But we will in fact prove more than what is stated in Theorem 1.5.

We assume that U_n and V_m are Haar spaces of dimension n and m, respectively, $n, m \geqslant 2$, in $C(B)$. (The cases where $n = 1$ or $m = 1$ are covered by Proposition 1.4.) Since B is compact and $C(B)$ contains a Haar space of dimension > 1, it follows from Mairhuber's Theorem (see Mairhuber [9]) that B is topologically imbeddable in S^1 (the circle in $\mathbb{R}^2$) and if n is even, this imbedding is into a strict subset of S^1. Our B is somewhat more specific. As such, topological imbeddability is equivalent to the existence of a homeomorphism (continuous one-to-one map) between the appropriate sets. This means that we may consider B as either a finite union of closed, disjoint intervals (none of which are singletons by our initial assumption) in $\mathbb{R}$, or as S^1, in which case both n and m are odd.

We first prove strengthened versions of Theorem 1.6.

THEOREM 3.1. *Let* B *be a finite union of closed, disjoint intervals of* $\mathbb{R}$ *(none of which are singletons). Assume* U_n *and* V_m *are* n*- and* m*-dimensional Haar spaces in* $C(B)$, *respectively*, $n, m \geqslant 2$, *and* $\dim(U_n V_m) = n + m - 1$. *Then we can write* U_n *and* V_m *in the form*

$$U_n = \text{span}\{w_1 h^{i-1} : i = 1, ..., n\} \tag{3.1}$$

$$V_m = \text{span}\{w_2 h^{i-1} : i = 1, ..., m\}, \tag{3.2}$$

where

(a) $w_1, w_2, h \in C(B)$.

(b) w_1, w_2 *never vanish on* B.

(c) h *is* 1–1 *on* B.

Note that we are claiming that it is possible to choose h without any singularity. This is not possible if $B = S^1$.

THEOREM 3.2. *Assume U_n and V_m are n- and m-dimensional Haar spaces in $C(S^1)$, respectively, n, m odd, $n, m \geqslant 2$, and $\dim(U_n V_m) = n + m - 1$. Then we can write U_n and V_m in the form*

$$U_n = \text{span}\{w_1 h^{i-1} : i = 1, ..., n\} \tag{3.3}$$

$$V_m = \text{span}\{w_2 h^{i-1} : i = 1, ..., m\}, \tag{3.4}$$

where

(a) $w_1, w_2 \in C(S^1)$.

(b) *There exists one point $x^* \in S^1$ such that $w_1(x^*) = w_2(x^*) = 0$, and w_1, w_2 are strictly positive at all other points of S^1.*

(c) *h is continuous and strictly increasing on $S^1 \backslash \{x^*\}$, and the range of h is all of $\mathbb{R}$; i.e., $\lim_{x \to x^{*-}} h(x) = \infty$, $\lim_{x \to x^{*+}} h(x) = -\infty$.*

Furthermore, up to multiplication by constants and the choice of x^*, properties (a), (b), *and* (c) *hold for all w_1, w_2 and h satisfying* (3.3) *and* (3.4).

Remark. Our abuse of mathematical precision in (c) should be understood thus. Let

$$S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\}$$

and $x^* = e^{i\theta^*}$. The function $h(e^{i\theta})$ is continuous and strictly increasing as a function of θ on $(\theta^*, \theta^* + 2\pi)$, and its range thereon is all of $\mathbb{R}$.

Proof of Theorem 3.1. Based on Theorem 1.6 and the Haar space property, we first prove some preliminary facts which will also be used in the proof of Theorem 3.2.

Assume $w_1(x^*) = 0$. Since U_n is a Haar space there must exist some $u \in U_n$ for which $u(x^*) \neq 0$. This implies, see the remark near the end of Section 2, that $(w_1 h^{i-1})(x^*) = 0$, $i = 1, ..., n-1$, and $(w_1 h^{n-1})(x^*) \neq 0$.

Thus $\lim_{x \to x^*} |h(x^*)| = \infty$, which in turn implies that $w_2(x^*) = 0$. Hence w_1 and w_2 share the same zero set.

Now assume $w_1(x^*) = w_1(\tilde{x}) = 0$ for some $x^* \neq \tilde{x}$. But then $(w_1 h^{i-1})(x^*) = (w_1 h^{i-1})(\tilde{x}) = 0$, $i = 1, ..., n-1$, which implies that

$$\dim U_n|_{\{x^*, \tilde{x}\}} = 1.$$

This contradicts the Haar space property of U_n. Thus w_1 and w_2 have at most one zero, and if it exists, it is a common zero.

Let $x_1, ..., x_n$ be n distinct points in B, not including x^* the zero of w_1, if such a point exists. Then from the Haar space property of U_n,

$$0 \neq \det((w_1 h^{i-1})(x_j))_{i, j=1}^n.$$

We can easily calculate the above Vandermonde type determinant. It equals

$$\left[\prod_{j=1}^n w_1(x_j)\right] \prod_{1 \leqslant j < k \leqslant n} (h(x_k) - h(x_j)).$$

Thus h is 1–1 on B.

Now let us assume that B is a finite union of closed disjoint intervals of $\mathbb{R}$ (none of which is a singleton). h is continuous on the set where w_1 (or w_2) does not vanish. Thus if w_1 does not vanish on B, then h is both continuous and 1–1 on B and Theorem 3.1 is proved. Assume there exists an $x^* \in B$ such that $w_1(x^*) = w_2(x^*) = 0$. We claim that the range of h cannot be all of $\mathbb{R}$. Since h is continuous and 1–1 on $B \backslash \{x^*\}$ it follows that on each disjoint closed interval of B, the range of h is a finite closed interval, except on the interval containing x^*. On that interval the range of h will be $(-\infty, a]$, or $[b, \infty)$, or $(-\infty, a] \cup [b, \infty)$ for some $a < b$. These intervals (ranges) must all be disjoint (since h is 1–1) and thus cannot cover all of $\mathbb{R}$.

Choose $d \notin \text{range } h$ and set

$$H(x) = \frac{1}{h(x) - d}, \qquad x \in B.$$

From Lemma 2.7, it follows that there exist W_1, W_2 such that

$$U_n = \text{span}\{W_1 H^{i-1} : i = 1, ..., n\}$$
$$V_m = \text{span}\{W_2 H^{i-1} : i = 1, ..., m\}.$$

At no point $\tilde{x} \in B$ does $\lim_{x \to \tilde{x}} |H(x)| = \infty$. Thus W_1 and W_2 do not vanish on B and (a), (b), and (c) necessarily hold. ∎

Proof of Theorem 3.2. From the proof of Theorem 3.1 we have that w_1, $w_2 \in C(S^1)$ have at most one (common) zero on S^1. If no such zero exists, then h is 1–1 and continuous on S^1. This is impossible since $h(0) = h(2\pi)$. Thus there must exist a point $x^* \in S^1$ at which $w_1(x^*) = w_2(x^*) = 0$. However, since there is only one such point in S^1, w_1, w_2 cannot change sign at this point; i.e., we may assume that both w_1 and w_2 are strictly positive at all other points. As h is 1–1 and continuous on $S^1 \backslash \{x^*\}$ and $\lim_{x \to x^*} |h(x^*)| = \infty$, property (c) must hold for h or $-h$.

The above properties hold (up to multiplication by a constant and the choice of x^*) for any w_1, w_2 and h satisfying (3.3) and (3.4). This proves Theorem 3.2. Note that we may, replacing h by

$$H(x) = \frac{1}{h(x) - d},$$

select the point $x^* \in S^1$ by an appropriate choice of $d \in \mathbb{R}$. ∎

Remark. Theorems 3.1 and 3.2 embody the cases where B is a compact set. However, this is not the only possible setting. For example, let $B = [0, 2\pi)$ and let U_n, $V_m \subset C[0, 2\pi]$ be n- and m-dimensional Haar subspaces on $[0, 2\pi)$, respectively, satisfying $u(0) = cu(2\pi)$ for all $u \in U_n$, and $v(0) = dv(2\pi)$ for all $v \in V_m$, c, $d \in \mathbb{R} \backslash \{0\}$. Assume $\dim(U_n V_m) = n + m - 1$. What can we say about U_n and V_m? (If $c = d = 1$, then we refer to Theorem 3.2.) It follows from the Haar space property that if $c > 0$, then n is odd, while if $c < 0$, then n is even. From Theorem 1.6, U_n and V_m have the form

$$U_n = \operatorname{span}\{w_1 h^{i-1} : i = 1, ..., n\}$$
$$V_m = \operatorname{span}\{w_2 h^{i-1} : i = 1, ..., m\}.$$

From an analysis similar to that in the above proofs of Theorems 3.1 and 3.2 one can prove that w_1, $w_2 \in C[0, 2\pi]$, $w_1(0) = cw_1(2\pi)$, $w_2(0) = dw_2(2\pi)$, and there exists exactly one point $x^* \in [0, 2\pi)$ for which $w_1(x^*) = w_2(x^*) = 0$ (this point may be chosen). If $x^* \in (0, 2\pi)$, then w_1, resp. w_2, does not change sign at x^* if $c > 0$, resp. $d > 0$, and does change sign at x^* if $c < 0$, resp. $d < 0$. h is continuous on $[0, 2\pi) \backslash \{x^*\}$ and may be chosen to be strictly increasing on $[0, 2\pi) \backslash \{x^*\}$. h also satisfies $h(0) = h(2\pi)$, and the range of h is all of $\mathbb{R}$.

Remark. From Theorems 3.1 and 3.2 it easily follows that if $u \in U_n$ and $v \in V_m$ have a common zero, then it can be factored out. (In the situation of Theorem 3.2 we can always assume that the common zero of u and v is not the common zero of w_1 and w_2 as this latter zero may be freely selected.) That is, if $u(\tilde{x}) = v(\tilde{x}) = 0$, then $h - h(\tilde{x})$ divides both u and v (and the numerator and denominator remain within U_n and V_m, respectively). This implies, see for example Cheney [4, Chap. 5, Sect. 2], that U_n/V_m is an existence set for $C(B)$. By that we mean that to every $f \in C(B)$ there exists a best approximant from U_n/V_m. This should be emphasized. In the Introduction (see Theorem 1.1 and Proposition 1.2) we always considered r^* a best approximant from U_n/V_m. We purposely did not consider the possibility that a best approximation exists from the (correct) closure of U_n/V_m, but not from U_n/V_m itself. This cannot occur here. The above form of U_n and V_m implies that every best approximant to any $f \in C(B)$ from the (correct) closure of U_n/V_m can in fact be written as an element of U_n/V_m (as long as in the situation of Theorem 3.2 we consider a form where the common zero of u and v is not the common zero of w_1 and w_2).

Proof of Theorem 1.5. We will prove that for each $r^* \in U_n/V_m$, the subspace $U_n + r^* V_m \subset C(B)$ is a Haar space. From Proposition 1.2 this proves the uniqueness property of U_n/V_m. We divide the proof into the two cases delineated by Theorems 3.1 and 3.2.

We first assume that the conditions of Theorem 3.1 hold; i.e., B is not homeomorphic to S^1, and U_n and V_m are as given in (3.1) and (3.2). This is the simpler case and we essentially follow the proof given in Cheney [4, Chap. 5, Sect. 3].

If $u = w_1(\sum_{i=1}^k a_i h^{i-1})$, $a_k \neq 0$, then we say u has *degree* $k-1$ and set $\partial u = k-1$. Thus, for example, $\partial(w_1) = 0$. We do the same for $v \in V_m$. (Set $\partial 0 = -\infty$ and by convention assume that if $r^* = u^*/v^* = 0$, then $\partial u^* = -\infty$ and $\partial v^* = 0$.)

We shall prove that with this notation, and for any $r^* = u^*/v^* \in U_n/V_m$ in irreducible form (no common factors of h), $U_n + r^* V_m$ is a Haar space of dimension

$$\max\{n + \partial v^*, m + \partial u\}.$$

The case $r^* = 0$ is trivial and as such we assume $r^* \neq 0$. We first prove the dimension formula. We have

$$\dim(U_n + r^* V_m) = \dim(U_n) + \dim(r^* V_m) - \dim(U_n \cap r^* V_m),$$

where $\dim(U_n) = n$, $\dim(r^* V_m) = m$. We must thus calculate $\dim(U_n \cap r^* V_m)$. Let $u^* = w_1(\sum_{i=1}^k a_i^* h^{i-1})$, $a_k^* \neq 0$, and $v^* = w_2(\sum_{i=1}^{\ell} b_i^* h^{i-1})$, $b_\ell^* \neq 0$, with no common factors. Thus $k = \partial u^* + 1$ and $\ell = \partial v^* + 1$. Now

$$r^* V_m = \left\{ \frac{u^*}{v^*} v : v \in V_m \right\}$$

$$= \left\{ \frac{w_1(\sum_{i=1}^k a_i^* h^{i-1})}{(\sum_{i=1}^{\ell} b_i^* h^{i-1})} \left(\sum_{i=1}^{m} c_i h^{i-1} \right) : c_1, ..., c_m \in \mathbb{R} \right\}.$$

As u^* and v^* have no common factors, in order for u^*v/v^*, $v \in V_m$, to be an element of U_n it is necessary (and sufficient) that v factor in the form

$$v = v^* \left(\sum_{i=1}^{s} d_i h^{i-1} \right)$$

with arbitrary $d_1, ..., d_s \in \mathbb{R}$ and with certain restrictions on s. What are these restrictions? A simple counting shows that in order for $v \in V_m$ we need $s \leqslant m - \partial v^*$, while in order that $u^*v/v^* \in U_n$ we need $s \leqslant n - \partial u^*$. Thus

$$\dim(U_n \cap r^* V_m) = \min\{m - \partial v^*, n - \partial u^*\}$$

and

$$\begin{aligned}\dim(U_n + r^* V_m) &= n + m - \min\{m - \partial v^*, n - \partial u^*\} \\ &= \max\{n + \partial v^*, m + \partial u^*\}.\end{aligned}$$

It remains to prove that $U_n + r^* V_m$ is a Haar space. This follows from the form of $U_n + r^* V_m$. For any $u \in U_n$, $v \in V_m$, the zero set of $u + r^*v$ is identical to that of $uv^* + u^*v$, since we have assumed that v^* does not vanish on B. Now

$$\begin{aligned} &uv^* + u^*v \\ &\quad = w_1 w_2 \left[\left(\sum_{i=1}^{n} c_i h^{i-1} \right) \left(\sum_{i=1}^{\ell} b_i^* h^{i-1} \right) + \left(\sum_{i=1}^{k} a_i^* h^{i-1} \right) \left(\sum_{i=1}^{m} d_i h^{i-1} \right) \right] \\ &\quad = w_1 w_2 \left[\sum_{i=1}^{s} \alpha_i h^{i-1} \right] \end{aligned}$$

where $s \leqslant \max\{n + \partial v^*, m + \partial u^*\} = \dim(U_n + r^* V_m)$. Since w_1, w_2 do not vanish on B, and h is 1–1 thereon, no nonzero function of the above form has more than $s - 1$ distinct zeros in B. Thus $U_n + r^* V_m$ is a Haar space.

We now assume that the conditions of Theorem 3.2 hold; i.e., B is homeomorphic to S^1, and U_n and V_m are as given in (3.3) and (3.4). This is the more interesting case.

As previously, we assume that $r^* = u^*/v^* \neq 0$ is in irreducible form (no common factors of h) and we will prove that $U_n + r^* V_m$ is a Haar space. Again we assume that v^* does not vanish on B. Previously this was well understood, and presented no problem. However as w_1 and w_2 vanish at some point, and $|h|$ tends to infinity at this same point, we should explain what is meant here. We simply assume that u^*/v^* is well defined (and thus finite) at each point of B. This imposes certain conditions.

Assume $w_1(x^*) = w_2(x^*) = 0$ and u^*/v^* does not vanish at x^*. Let

$$u^* = w_1\left(\sum_{i=1}^{k} a_i^* h^{i-1}\right), \qquad a_k^* \neq 0$$

and

$$v^* = w_2\left(\sum_{i=1}^{\ell} b_i^* h^{i-1}\right), \qquad b_\ell^* \neq 0.$$

Then

$$\lim_{x \to x^*} \frac{u^*(x)}{v^*(x)} = \lim_{x \to x^*} \frac{w_1(x)\, a_k^* h^{k-1}(x)}{w_2(x)\, b_\ell^* h^{\ell-1}(x)} = \frac{a_k^*}{b_\ell^*} \lim_{x \to x^*} \frac{w_1(x)}{w_2(x)} h^{k-\ell}(x).$$

From (3.3) and (3.4), $\lim_{x \to x^*} w_1(x)\, h^{n-1}(x)$ and $\lim_{x \to x^*} w_2(x)\, h^{m-1}(x)$ both exist and are non-zero. Thus

$$\lim_{x \to x^*} \frac{w_1(x)}{w_2(x)} h^{n-m}(x)$$

exists and is nonzero. As we assume that

$$\lim_{x \to x^*} \frac{u^*(x)}{v^*(x)}$$

exists and is nonzero, we must have $k - \ell = n - m$. If u^*/v^* vanishes at x^*, then we obtain $k - \ell \leqslant n - m$. In either case $m + k \leqslant n + \ell$, i.e., $m + \partial u^* \leqslant n + \partial v^*$.

In addition,

$$v^* = w_2\left(\sum_{i=1}^{\ell} b_i^* h^{i-1}\right) = w_2 \beta \prod_{j=1}^{\ell-1} (h - \gamma_j) \tag{3.5}$$

for some $\beta \neq 0$. As the range of h is all of $\mathbb{R}$, it follows that if γ_j is real, then v^* has a zero in B at some point $\tilde{x}$, other than x^*, where $h(\tilde{x}) = \gamma_j$. For u^*/v^* to be well-defined and finite at $\tilde{x}$ it is therefore necessary that $u^*(\tilde{x}) = 0$. But then $(h - \gamma_j)$ is a common factor of u^* and v^* contradicting our assumption that they have no common factors. (We also contradict our assumption that v^* does not vanish on B.) This implies that each γ_j in (3.5) is in $\mathbb{C} \backslash \mathbb{R}$. As the b_i^* are real, these nonreal roots of v^* come in complex conjugate pairs. Thus

$$v^* = w_2 \beta \prod_{j=1}^{s} (h - \delta_j)(h - \bar{\delta}_j)$$

for $\delta_j \notin \mathbb{R}$. Therefore $\ell - 1 = 2s$, i.e., $\partial v^* = \ell - 1 = 2s$ (is necessarily even) and hence $n + \partial v^*$ is odd.

With the above facts we can now parallel the previous proof and show that $U_n + r^* V_m$ is a Haar space of dimension $n + \partial v^*$. ∎

4. TRIGONOMETRIC POLYNOMIALS

Let

$$T_n = \operatorname{span}\{1, \sin x, \cos x, ..., \sin nx, \cos nx\}.$$

T_n is a periodic Haar space of dimension $2n+1$ on $[0, 2\pi)$. It is the prototype of a Haar space on B, where B is homeomorphic to S^1. T_n (together with T_m) satisfy the assumptions of Theorem 3.2. If $w(x) = 1 - \cos x$ and $h(x) = \sin x/(1 - \cos x)$, then it may be easily calculated that

$$T_n = \operatorname{span}\{w^n h^{i-1} : i = 1, ..., 2n+1\}. \tag{4.1}$$

This is a non-standard (but useful) basis for T_n. It may also be rewritten as follows. Recall that $1 - \cos x = 2\sin^2(x/2)$ and $\sin x = 2\sin(x/2)\cos(x/2)$. Thus

$$h(x) = \frac{\sin x}{1 - \cos x} = \frac{\cos(x/2)}{\sin(x/2)}$$

and up to a constant

$$w^n h^{i-1} = (\sin(x/2))^{2n-i+1} (\cos(x/2))^{i-1}, \qquad i = 1, ..., 2n+1,$$

i.e.,

$$T_n = \operatorname{span}\{(\sin(x/2))^{2n-i} (\cos(x/2))^i : i = 0, 1, ..., 2n\}.$$

For $t \in T_p \backslash T_{p-1}$, let $\tilde{\partial} t = 2p$. (Set $\tilde{\partial} 0 = -\infty$, $\tilde{\partial} 1 = 0$.). This differs from the ∂t as defined in the proof of Theorem 1.5, and it is this difference which we now discuss. In Lorentz *et al.* [8, p. 217] it is proven that for $r^* = u^*/v^* \in T_n/T_m$

$$\dim(T_n + r^* T_m) = \max\{2n + 1 + \tilde{\partial} v^*, 2m + 1 + \tilde{\partial} u^*\}.$$

On the other hand we have proved that

$$\dim(T_n + r^* T_m) = 2n + 1 + \partial v^*,$$

(and $2n+1+\partial v^* \geqslant 2m+1+\partial u^*$). It must be that these two quantities are the same. But why?

One explanation is the following. Assume that we have chosen the basis

$$T_n = \text{span}\{w^n h^{i-1} : i=1, ..., 2n+1\},$$

where the w and h are as in (4.1). We further assume, without great loss of generality, that $r^*(0) \neq 0$ (here $x^*=0$).

Let

$$u^* = w^n \left(\sum_{i=1}^{k} a_i^* h^{i-1} \right), \qquad a_k^* \neq 0$$

$$v^* = w^m \left(\sum_{i=1}^{\ell} b_i^* h^{i-1} \right), \qquad b_\ell^* \neq 0.$$

It was shown in the proof of Theorem 1.5 that $k-\ell = 2n-2m$, and $\partial v^* = \ell - 1 = 2s$. Thus $\partial u^* = k-1 = 2q$ and $2n+1+\partial v^* = 2m+1+\partial u^*$. (If $r^*(0)=0$, then we obtain $2n+1+\partial v^* \geqslant 2m+1+\partial u^*$.)

Every $t \in T_p$ may be written in the form

$$t = w^p \sum_{i=1}^{2p+1} c_i^* h^{i-1}. \tag{4.2}$$

Moreover since $t \in T_n$ for every $n > p$, it may also be written in the form

$$t = w^n \sum_{i=1}^{2n+1} c_{i,n}^* h^{i-1}, \tag{4.3}$$

for some unique choice of $c_{i,n}^*$. How are the forms (4.2) and (4.3) related?

It is easily checked that

$$1 = \frac{w}{2}(1+h^2).$$

Thus from the uniqueness of the coefficients in (4.2) and (4.3) we must have (as a function of h)

$$\sum_{i=1}^{2n+1} c_{i,n}^* h^{i-1} = \left(\sum_{i=1}^{2p+1} c_i^* h^{i-1} \right) \left(\frac{1+h^2}{2} \right)^{n-p}$$

for each $n > p$. This is how (4.2) and (4.3) are related.

We now return to u^* and v^*. By assumption u^* and v^* have no common factors of h. From the above analysis it therefore follows that either $\partial u^* = \tilde{\partial} u^*$ and $\partial v^* \geqslant \tilde{\partial} v^*$, or $\partial u^* \geqslant \tilde{\partial} u^*$ and $\partial v^* = \tilde{\partial} v^*$. (The only other option is that $\partial u^* > \tilde{\partial} u^*$ and $\partial v^* > \tilde{\partial} v^*$ in which case both u^* and v^* contain common positive powers of $(1+h^2)$.) Furthermore we recall that $2n+1+\partial v^* = 2m+1+\partial u^*$. Thus

$$\max\{2n+1+\tilde{\partial} v^*, 2m+1+\tilde{\partial} u^*\} = 2n+1+\partial v^*.$$

If $r^*(0)=0$, then the same final result holds.

Remark. The space

$$C_n = \operatorname{span}\{1, \cos x, ..., \cos nx\}$$

has the form

$$C_n = \operatorname{span}\{wh^{i-1} : i=1, ..., n+1\},$$

where $w=1$ and $h=\cos x$. It is a Haar space of dimension $n+1$ on $[0, \pi]$. The space

$$S_n = \operatorname{span}\{\sin x, ..., \sin nx\}$$

also has the form

$$S_n = \operatorname{span}\{wh^{i-1} : i=1, ..., n\}.$$

Here $w=\sin x$ and $h=\cos x$. It is a Haar space of dimension n on $(0, \pi)$. Note that since C_n and S_m share a common h we have $\dim(C_n S_m) = n+m$. It follows (most easily using the above bases) that $C_n S_m = S_{n+m}$. Similarly $C_n C_m = C_{n+m}$, while $S_n S_m = (\sin x)\, S_{n+m-1}$; i.e.,

$$S_n S_m = \operatorname{span}\{\sin x \cdot \sin kx : k=1, ..., n+m-1\}.$$

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